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A NEW CLASS OF DISTRIBUTED-ELEMENT MODELS FOR CYCLIC PLASTICITY—II. ON IMPORTANT PROPERTIES OF MATERIAL BEHAVIOR

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Abstract—A general theory is developed to elucidate important properties of inelastic material behavior under cyclic multi-axial loading conditions based on the behavior of the new Distributed-Element Models proposed in Part I. By establishing some important theorems on the behavior of the DEMs, the property of *erasure-of-memory* exhibited by real materials is shown to be closely related to the existence and uniqueness of *equilibrium states*, which is in turn a consequence of the property that the admissible stress region bounded by the *limit surface* associated with a Distributed-Element Model is bounded and strictly convex.

1. INTRODUCTION

It is well recognized that plastic deformation of materials or structural systems is in general history dependent, that is, materials have a certain memory of the history of plastic deformation experienced. However, under some circumstances, the plastic response of a system may become independent of part of its previous deformation history, so that it seems that the past history has been erased from the memory of the system. This property of material systems has been observed experimentally and is generally referred to as the property of "erasure-of-memory' (Lamba and Sidebottom, 1978a, b). Although this property is useful in conducting experimental studies on cyclic plasticity, as far as we can ascertain, it has not yet been studied from a theoretical point of view.

In the simulation studies discussed in Part I (Chiang and Beck, 1993), we have already seen the existence of equilibrium points corresponding to different uni-directional strain paths, as well as the existence of a limit surface, exhibited by a new multi-dimensional DEM. In addition, the property of erasure-of-memory exhibited by real materials is demonstrated by the new model which gave excellent results for response predictions when compared to experimental observations. It is of great interest to further investigate these general properties of material behavior from a theoretical point of view, to better understand the complicated response behavior of cyclic plasticity. A thorough understanding of these properties may also provide useful insight and guidelines for validating analytical plasticity models and for performing analytical and experimental studies in the related areas of plasticity.

2. GENERAL TREATMENT OF INCREMENTAL THEORY OF PLASTICITY

In view of the physically consistent behavior of the multi-dimensional DEM shown in Part I, we would like to further study some relevant properties of the new model. To begin with, a general treatment of the incremental theory of plasticity is presented, and the general formulation is then used to derive important properties associated with the new class of multi-dimensional DEMs for cyclic plasticity.

Let $\tilde{\sigma} \equiv (\sigma_{ij})$ be the total stress tensor and $\tilde{\varepsilon} \equiv (\varepsilon_{ij})$ the total strain tensor at a point in a material. Define the *elastic* component and the *plastic-relaxation* component of the stress increment tensor, $d\tilde{\sigma}^e$ and $d\tilde{\sigma}^p$, respectively, by:

$$d\sigma_{ij}^{e} \equiv C_{ijkl}^{e}\, d\varepsilon_{kl}, \tag{1}$$

$$d\sigma_{ij}^{p} \equiv d\sigma_{ij}^{e} - d\sigma_{ij}, \tag{2}$$

where C_{ijkl}^{e} is an elastic modulus tensor which is assumed to be independent of response states.

Introduce a *plastic modulus reduction tensor* Λ_{ijkl} so that

$$d\sigma_{ij}^{p} \equiv C_{ijmn}^{e}\Lambda_{mnkl}(\tilde{\sigma}, \tilde{\varepsilon}, d\tilde{\sigma}, d\tilde{\varepsilon})\, d\varepsilon_{kl}, \tag{3}$$

then following from the incremental stress–strain relation:

$$d\sigma_{ij} = C_{ijkl}^{e}\, d\varepsilon_{kl}^{e} = C_{ijkl}^{e}(d\varepsilon_{kl} - d\varepsilon_{kl}^{p}), \tag{4}$$

where $d\varepsilon_{kl}^{e}$ and $d\varepsilon_{kl}^{p}$ are the elastic and plastic strain increment tensors, respectively, we can derive that

$$d\sigma_{ij}^{p} = C_{ijkl}^{e}\, d\varepsilon_{kl}^{p}$$

and

$$d\varepsilon_{ij}^{p} = \Lambda_{ijkl}(\tilde{\sigma}, \tilde{\varepsilon}, d\tilde{\sigma}, d\tilde{\varepsilon})\, d\varepsilon_{kl}. \tag{5}$$

Note that the plastic modulus reduction tensor is, in general, a function of not only the current state at a point in the material, but also of the load increment. It is zero if the material is in a purely elastic state at the point.

The above equations can be put in a vector form as follows. Since σ_{ij} and ε_{ij} are symmetric second-order tensors, they can be written as vectors $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \Re^6$ defined by

$$\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{13}]^T, \tag{6}$$

$$\boldsymbol{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{23}, 2\varepsilon_{13}]^T, \tag{7}$$

so that the values of the inner products between tensors and between vectors are preserved, and where the superscript T denotes matrix transpose, i.e.

$$\begin{aligned} \tilde{\sigma} \cdot \tilde{\varepsilon} &\equiv \sigma_{ij}\varepsilon_{ij} \\ &= \sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\sigma_{12}\varepsilon_{12} + 2\sigma_{23}\varepsilon_{23} + 2\sigma_{13}\varepsilon_{13} \\ &= \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}. \end{aligned}$$

Thus, eqn (5) becomes

$$d\boldsymbol{\varepsilon}^{p} = \boldsymbol{\Lambda}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, d\boldsymbol{\sigma}, d\boldsymbol{\varepsilon})\, d\boldsymbol{\varepsilon}, \tag{8}$$

and eqn (4) can be rewritten as

$$d\boldsymbol{\sigma} = \mathbf{C}^{e}[\mathbf{I} - \boldsymbol{\Lambda}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, d\boldsymbol{\sigma}, d\boldsymbol{\varepsilon})]\, d\boldsymbol{\varepsilon}, \tag{9}$$

where $\boldsymbol{\Lambda}$ and $\mathbf{C}^{e}$ are the matrices corresponding to the fourth-order tensors Λ_{ijkl} and C_{ijkl}^{e} so that the equations defined accordingly are consistent, and $\mathbf{I}$ is the 6×6 identity matrix. The elastic modulus matrix $\mathbf{C}^{e}$ is symmetric because of the symmetries associated with the tensor C_{ijkl}^{e} for elastic behavior. It is positive definite if the material is stable to small strain perturbations (or if Drucker's postulates hold), which we assume is the case. Equation (9) can be reformulated as

$$d\boldsymbol{\sigma} = [\mathbf{C}^e - \mathbf{C}^p(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, d\boldsymbol{\sigma}, d\boldsymbol{\varepsilon})]\, d\boldsymbol{\varepsilon}, \tag{10}$$

where $\mathbf{C}^p \equiv \mathbf{C}^e\boldsymbol{\Lambda}$ can be referred to as the *plastic modulus matrix.*

It can be shown (Chiang, 1992) that within the classical formulation of plasticity,† the plastic modulus reduction matrix $\boldsymbol{\Lambda}$ and the plastic modulus matrix $\mathbf{C}^p$ are both of rank one or zero, corresponding to yielding and elastic behavior, respectively. Also, if an associated flow rule is used, the plastic modulus matrix $\mathbf{C}^p$ is symmetric and the plastic modulus reduction matrix $\boldsymbol{\Lambda}$ has, at most, one nonzero positive eigenvalue whose value is never greater than one. Furthermore, in general, Drucker's postulates of stability imply that $\mathbf{C}^p$ is positive semi-definite.

Equation (9) [or (10)] gives the general formulation of the basic constitutive law that we will use in the following, where we derive important properties associated with the new class of DEMs for plasticity. In order to develop a mathematically rigorous theory on the properties of the DEM, we make the following definition regarding the "state of equilibrium";

Definition 1. *An "equilibrium point (state)" is a stress state associated with a uni-directional strain increment* $d\boldsymbol{\varepsilon} = \mathbf{c}\, dt$, *with* $\mathbf{c} \neq 0$ *and* $dt > 0$, *at which*

$$d\boldsymbol{\sigma} = C^e[\mathbf{I} - \boldsymbol{\Lambda}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, d\boldsymbol{\sigma}, d\boldsymbol{\varepsilon})]\, d\boldsymbol{\varepsilon} = \mathbf{0}, \tag{11}$$

i.e.

$$\boldsymbol{\Lambda}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \mathbf{0}, \mathbf{c})]\mathbf{c} = \mathbf{c}, \tag{12}$$

where the dependence of Λ *on* dt *is dropped since we are mainly concerned with rate-independent plasticity in the present study.*

The direction of the uni-directional strain increment, $\mathbf{c}$, that defines an equilibrium point is referred to as the *reference path* associated with that equilibrium point. From eqn (12), the reference path is an eigenvector with eigenvalue unity corresponding to Λ evaluated at the equilibrium state. The term "equilibrium point (state)" was introduced by Lamba and Sidebottom (1978a) when they observed the phenomenon that stress increments approach zero for appreciable strain increments in their biaxial loading tests of thin-walled copper tubes.

To demonstrate the concept of equilibrium points, we consider in the following a simple example where a proportional strain loading path is prescribed, as the path 0–1 shown in Fig. 1(a). The response of an element having perfect plastic behavior to such a loading path can be depicted in a corresponding element stress space as shown in Fig. 1(b). If the element yields at point P on path 0–1, then the corresponding stress state will just reach the yield surface at, say P' in the stress space. It can be shown (Chiang, 1992) that as the loading is continued, the stress state will move around on the yield surface in the direction of $P'Q'$ and will finally reach the equilibrium state at Q' and stop there, as shown schematically in the figure.

One important property associated with equilibrium states follows directly from the definition and can be stated as follows.

Theorem 1. *At an equilibrium state for a DEM (or a classical plasticity model employing an associated flow rule), the plastic modulus reduction matrix* $\boldsymbol{\Lambda}$ *is of rank one, and the only nonzero eigenvalue of* $\boldsymbol{\Lambda}$ *has a value of one.*

The proof of the theorem utilizes the aforementioned properties of the classical theory of plasticity and can be found in Chiang (1992). It can be deduced from the theorem that

†By classical formulation of plasticity we mean that the elastic–plastic response behavior is characterized by a yield condition, a flow rule, and a strain hardening rule. The flow rule relates the increment of plastic strain to the current state and the stress increment. The strain hardening rule specifies how the yield surface is changed during plastic flow.

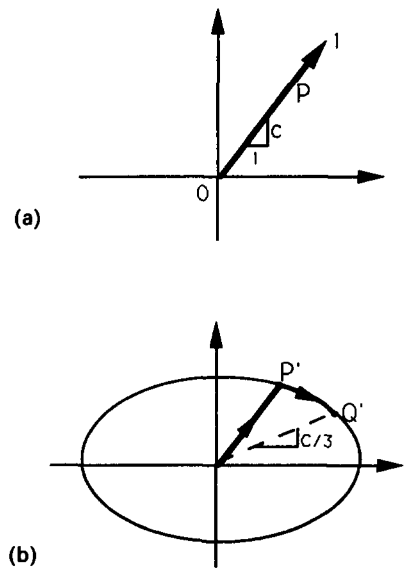

Fig. 1. An illustrative example to demonstrate the concept of equilibrium states: (a) Prescribed uni-directional strain path; (b) Stress response of elasto–perfectly plastic behavior.

the strain hardening effect must vanish as an equilibrium state is approached in the classical model. Furthermore, Theorem 1 implies a corollary that purely plastic incremental deformation of a DEM occurs only in a one-dimensional subspace of the six-dimensional strain-increment space if and only if the state is an equilibrium point. This property of the DEM differs from those of the models based on the classical theory of plasticity, by which purely plastic deformation always occurs in a one-dimensional subspace due to the use of the principle of normality (Chiang, 1992).

It should be noted that, since an equilibrium point is defined to be associated with a uni-directional strain *increment*, a strain cycle which is sufficiently smooth and long may also have particular equilibrium points associated with it. This situation is illustrated in Figs 2(a) and (b), where the ellipse in the ε–γ strain space denotes the prescribed strain cycle with discrete increments of constant magnitude, and the corresponding stress response calculated for a DEM shows two equilibrium points on the ellipse in the σ–τ stress space where the densest stress increments occur. This phenomenon leads to the so-called property of erasure-of-memory, since everytime the strain cycle is followed, the model is always brought back to the same stress state regardless of what the previous response history is. This will be discussed further when we introduce the concept of the limit surface associated with a DEM.

Two important issues are the existence and uniqueness of an equilibrium point given a specified reference path. Mathematically, it is difficult to show directly the existence of an equilibrium state considering the rather complicated formulation of plasticity involved. Instead, we use some simple energy arguments as presented in the following theorem.

Theorem 2. *For stable materials which have bounded elastic strain energy, given a specified uni-directional strain path, a corresponding equilibrium point always exists.*

Proof. Define the elastic strain energy of a system as

$$W^e \equiv \tfrac{1}{2}\varepsilon^e_{ij}C^e_{ijkl}\varepsilon^e_{kl} = \tfrac{1}{2}(\boldsymbol{\varepsilon}^e)^T\mathbf{C}^e\boldsymbol{\varepsilon}^e. \quad (13)$$

By assumption, W^e is bounded so that the elastic strain response ε^e_{ij} is also bounded. Along a uni-directional strain loading path $\boldsymbol{\varepsilon} = \mathbf{c}t \neq \mathbf{0}$, where, by definition, t is monotonically increasing ($dt > 0$), the elastic strain energy would never decrease after a certain state at, say, $t = t_i$. Thus, it requires for all $t > t_i$:

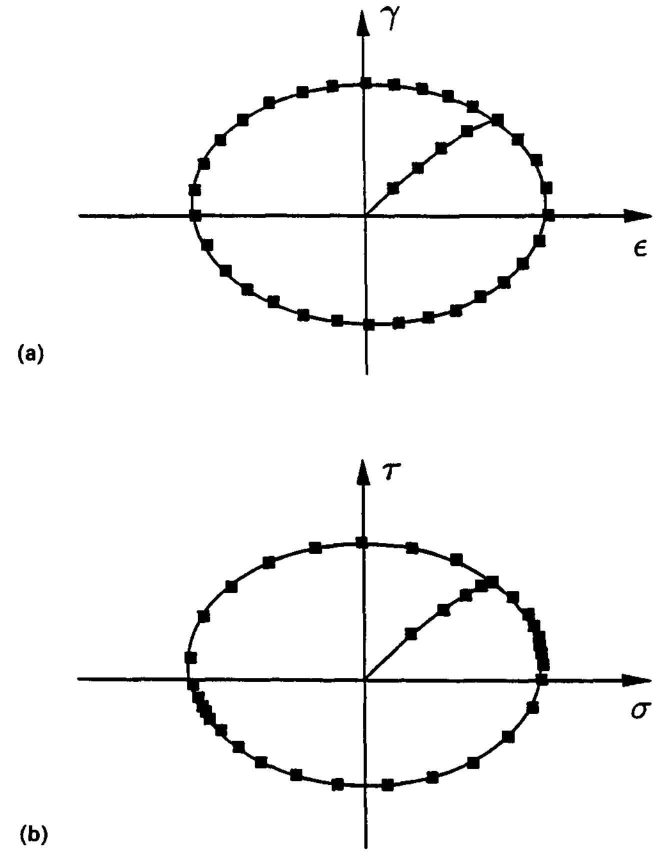

Fig. 2. Illustration of existence of the equilibrium points associated with a big strain cycle: (a) a big strain cycle; (b) the corresponding stress response.

$$\mathrm{d}W^e \approx \varepsilon^e_{ij} C^e_{ijkl}\,\mathrm{d}\varepsilon^e_{kl} = (\boldsymbol{\varepsilon}^e)^T \mathbf{C}^e\,\mathrm{d}\boldsymbol{\varepsilon}^e \geqslant 0. \tag{14}$$

For bounded elastic strain energy we must have $\mathrm{d}W^e(t) \to 0$, as $t \to t_0$, where t_0 corresponds to some state at which $\mathrm{d}W^e(t) = 0$, though possibly $t_0 = \infty$. By (14), we must have†

$$\mathrm{d}\boldsymbol{\varepsilon}^e(t_0) = \mathbf{0},$$

i.e.

$$\mathrm{d}\boldsymbol{\sigma}(t_0) = \mathbf{C}^e\,\mathrm{d}\boldsymbol{\varepsilon}^e(t_0) = \mathbf{0}. \tag{15}$$

Thus, it follows from the definition of an equilibrium point that the existence of an equilibrium state associated with a reference path $\mathrm{d}\boldsymbol{\varepsilon} = \mathbf{c}\,\mathrm{d}t \neq \mathbf{0}$ is always guaranteed.

The issue of uniqueness of an equilibrium point associated with a reference path will be discussed later after we have introduced the concept of the limit surface and its associated properties.

3. GEOMETRIC CONSIDERATION OF YIELD SURFACES FOR THE NEW DEM

In defining the kinematics of an element in the DEM, we introduced a yield condition for characterizing the general multi-axial elasto–plastic behavior of the element. The yield condition has been defined in the same sense as has been done in the classical theory of plasticity. In other words, the yield condition for a given material is essentially the extension of a single yield point in the uniaxial (or one-dimensional) case to a hypersurface in the six-

†We can rule out the possibility that $\mathrm{d}\boldsymbol{\varepsilon}^e$ becomes orthogonal to $\mathbf{C}^e\boldsymbol{\varepsilon}^e$, since in (14), we have neglected the high-order terms, which never vanish unless $\mathrm{d}\boldsymbol{\varepsilon}^e = \mathbf{0}$.

dimensional stress space (considering the symmetry of stress tensors). Since the DEM consists only of ideal plasticity elements, we may concentrate on the corresponding formulation, so that a yield condition is simply described by

$$F(\sigma_{ij}(k), k) = 0, \tag{16}$$

where k represents a yield constant corresponding to some particular element. For isotropic materials, since rotating the axes does not affect the yielding behavior, we can choose the principal stress axes for defining the coordinate system so that eqn (16) may be rewritten as

$$F(\sigma_1(k), \sigma_2(k), \sigma_3(k), k) = 0. \tag{17}$$

In the $(\sigma_1, \sigma_2, \sigma_3)$ coordinate system, which represents a stress space sometimes referred to as the Haigh–Westergaard stress space, eqn (17) specifies a normal three-dimensional surface that one can easily picture. In the following, we will formulate some important properties related to the yield surfaces of a DEM based on some basic principles in operator theory.

Recall that in Part I we defined a DEM as consisting of a collection of elasto–perfectly-plastic elements whose yield surfaces are nested within one another and are governed by yield functions of the same mathematical form so that the yield surfaces may have similar shapes. To make it clearer, we introduce the following definition.

Definition 2. Two hypersurfaces S_1: $F(\boldsymbol{\sigma}, k_1) = 0$ and S_2: $F(\boldsymbol{\sigma}, k_2) = 0$ are said to be similar *(in shape) with dimension ratio k_1/k_2, if any ray from the origin that passes through S_1 at $\boldsymbol{\sigma}_1$ intersects S_2 at $\boldsymbol{\sigma}_2$ such that*

$$k_2\boldsymbol{\sigma}_1 = k_1\boldsymbol{\sigma}_2.$$

Mathematically, if the dimension ratio of two similar surfaces S_2 and S_1 is $c > 0$, then by definition we have

$$F(\boldsymbol{\sigma}_0, k_0) = 0 \Leftrightarrow F(c\boldsymbol{\sigma}_0, ck_0) = 0. \tag{18}$$

Thus, we may infer the following theorem regarding the condition for similar surfaces.

Theorem 3. A set of yield surfaces S defined by $S = \{\boldsymbol{\sigma}: F(\boldsymbol{\sigma}, ck_0) = 0, c > 0\}$ are all similar with dimensions proportional to c, if the yield function $F(\ldots, \ldots)$ is homogeneous (of any order).

Proof. If $F(\ldots, \ldots)$ is homogeneous of some order, say m, and

$$F(\boldsymbol{\sigma}_0, k_0) = 0,$$

then

$$F(c\boldsymbol{\sigma}_0, ck_0) = c^m F(\boldsymbol{\sigma}_0, k_0) = 0 \quad \forall\ c > 0.$$

By eqn (18), we may conclude that all surfaces are similar with dimensions proportional to c.

Based on the above result, we now assume that the yield function used to define the yield surfaces of a DEM is homogeneous so that the nested yield surfaces are all similar in shape with dimensions proportional to the yield constants k. Thus, the domain of elasticity, Ω_i, in the element stress space for an element with yield constant k_i, defined by

$$\Omega_i = \{\sigma_i : F(\sigma_i, k_i) < 0\}, \tag{19}$$

can be expressed as

$$\Omega_i = k_i \Omega_0 \tag{20}$$

or

$$\Omega_i = \{k_i \sigma_0^{(i)} : \sigma_0^{(i)} \in \Omega_0\}, \tag{21}$$

where Ω_0 is the domain of elasticity of some element with yield constant $k_0 = 1$, and it is a bounded, convex set. The boundedness follows from the fact that any real material has finite ultimate strength (peak stress), and the convexity follows from the well-known result that a yield surface is convex if Drucker's postulates hold (Mendelson, 1968), as we assume. Since the model response of a DEM can be written as (using the formulation of a finite number of elements):

$$\sigma = \sum_{i=1}^{N} \psi_i \sigma_i, \tag{22}$$

where N is the total number of elements and

$$\sum_{i=1}^{N} \psi_i = 1, \psi_i \geqslant 0, \tag{23}$$

by operator theory on convex sets (Kadison and Ringrose, 1983), the set of all model stress points, denoted as the domain Ω, is given by

$$\Omega = \sum_{i=1}^{N} (\psi_i \Omega_i) = \sum_{i=1}^{N} (\psi_i k_i \Omega_0) = \left(\sum_{i=1}^{N} \psi_i k_i\right) \Omega_0 \quad (\text{since } \Omega_0 \text{ is convex and } k_i \psi_i > 0)$$
$$= k_u \Omega_0 \quad \left(k_u \equiv \sum_{i=1}^{N} \psi_i k_i\right). \tag{24}$$

By (24), the existence of Ω is guaranteed for finite N and also in the case where $N \to \infty$ as long as $k_u < \infty$, which may again be thought of as a condition of finite ultimate strength for any real materials. Furthermore, we may conclude that Ω is similar in shape to Ω_0. Thus, the boundary of Ω, $\partial\Omega$, defines a limit surface of a DEM, which can be described by

$$F(\sigma, k_u) = 0, \tag{25}$$

such that a model stress state can never go beyond the limit surface associated with the model. This proves the following theorem specifying an important property of the DEM:

Theorem 4. *There exists a limit surface associated with a DEM, described by*

$$F(\sigma, k_u) = 0,$$

where $k_u \equiv \Sigma_{i=1}^{N} \psi_i k_i$, and k_i, $i = 1, \ldots, N$, are the yield constants of the N elements constituting the model. The limit surface is similar in shape to the yield surfaces associated with each of the distributed elements if the yield function employed is homogeneous.

In the following, we will derive some important properties related to the equilibrium points and the limit surface associated with a DEM. First of all, we note that, from Definition 1, at an equilibrium state corresponding to a reference path $d\varepsilon = \mathbf{c}\, dt$, the plastic–strain response increment will be the same as the prescribed strain increment, i.e.

$$d\boldsymbol{\varepsilon}^{p}(\boldsymbol{\sigma}_{eq}^{c}) = \boldsymbol{\Lambda}(\boldsymbol{\sigma}_{eq}^{c}, d\boldsymbol{\varepsilon})\, d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon} = \mathbf{c}\, dt.$$

Thus, we have the following theorem pertaining to the equilibrium states of a DEM.

Theorem 5. At an equilibrium state of a DEM, all elements in the model are in corresponding equilibrium states, which lie on the associated yield surfaces at points having the same outward normal direction as the reference strain path, and conversely (all elements are in equilibrium state implies DEM is in an equilibrium state).

Proof. The converse is trivial since if each element is in an equilibrium state, we have $\forall\ i\ d\boldsymbol{\sigma}_i = 0$ for $d\boldsymbol{\varepsilon} = \mathbf{c}\, dt$, the common total strain increment, then $d\boldsymbol{\sigma} = \Sigma_{i=1}^{N} \psi_i\, d\boldsymbol{\sigma}_i = 0$.

Now, if a DEM is in an equilibrium state corresponding to a reference path $\mathbf{c}$, then the work done by $d\boldsymbol{\sigma}$ over any strain loading increment $d\boldsymbol{\varepsilon} = \mathbf{c}\, dt$ must be zero, since the corresponding stress increment vanishes. Since the DEM actually consists of an assemblage of ideal plasticity elements that are subject to the same total strain increment, the sum of the work done by all the elements must also vanish. Thus, since the incremental work done by each individual element is non-negative (Drucker's postulate), we may conclude that

$$d\boldsymbol{\sigma}_i^{T}\, d\boldsymbol{\varepsilon} = 0 \quad \forall\ i = 1, \ldots, N. \tag{26}$$

Also, by the assumption of ideal plasticity for each element, we have (Mendelson, 1968)

$$d\boldsymbol{\sigma}_i^{T}\, d\boldsymbol{\varepsilon}_i^{p} = 0 \quad \forall\ i = 1, \ldots, N. \tag{27}$$

It follows from (26) and (27) that

$$d\boldsymbol{\sigma}_i^{T}\, d\boldsymbol{\varepsilon}_i^{e} = (d\boldsymbol{\varepsilon}_i^{e})^{T}\mathbf{C}^{e}\, d\boldsymbol{\varepsilon}_i^{e} = 0 \quad \forall\ i = 1, \ldots, N.$$

Since $\mathbf{C}^{e}$ is positive definite, we must have $d\boldsymbol{\varepsilon}_i^{e} = \mathbf{0} \quad \forall\ i$, and hence $d\boldsymbol{\sigma}_i = \mathbf{0} \quad \forall\ i$, which shows that each element is in an equilibrium state corresponding to reference path $\mathbf{c}$. Furthermore, each $d\boldsymbol{\varepsilon}_i^{p} = d\boldsymbol{\varepsilon} = \mathbf{c}\, dt$, so by the principle of normality for each element, the outward normal at each element's equilibrium point is in the direction of $\mathbf{c}$.

Theorem 6. If all the element stress states of a DEM lie on the associated yield surfaces and line up in the stress space on a ray from the origin, then the stress state of the DEM is on the associated limit surface.

Proof. Firstly, we note that each yield surface associated with an element is the boundary, $\partial\Omega_i$, of the domain of elasticity Ω_i of that element, i.e.

$$\partial\Omega_i = \{\boldsymbol{\sigma}_i : F(\boldsymbol{\sigma}_i, \mathbf{k}_i) = 0\}. \tag{28}$$

From eqn (20), we have

$$\partial\Omega_i = k_i(\partial\Omega_0), \tag{29}$$

that is, each yield surface is described by

$$\partial\Omega_i = \{k_i\boldsymbol{\sigma}_0 : \boldsymbol{\sigma}_0 \in \partial\Omega_0\}. \tag{30}$$

Also, from Theorem 4, the limit surface is the set given by

$$\partial\Omega = \{k_u\boldsymbol{\sigma}_0 : \boldsymbol{\sigma}_0 \in \partial\Omega_0\}. \tag{31}$$

Thus, if all the element stress states of a DEM lie on the associated yield surfaces and line up in the stress space on a ray from the origin, then the element stress states must be proportional to one another with proportionality constants of yield strengths, i.e.

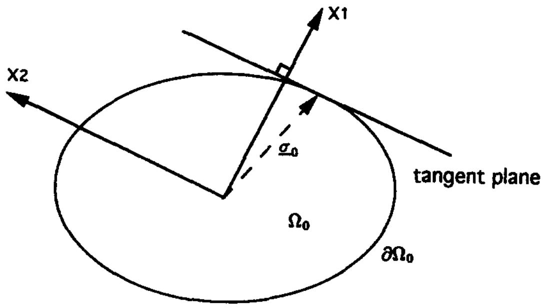

Fig. 3. A diagram showing the rotation of coordinate axes which makes the x_1 axis perpendicular to the tangent plane to the yield surface $\partial\Omega_0$ at σ_0.

$$\sigma_i = k_i\sigma_0, \quad (\text{for some } \sigma_0 \in \partial\Omega_0)$$

and hence we have

$$\sigma \equiv \sum_{i=1}^{N} \psi_i\sigma_i = \sum_{i=1}^{N} \psi_i(k_i\sigma_0)$$
$$= \left(\sum_{i=1}^{N} k_i\psi_i\right)\sigma_0 = k_u\sigma_0.$$

From eqn (31), the conclusion of the theorem follows.

It is of great importance to note that the limit surface of a DEM is like the yield surface of a model of ideal plasticity as far as the plastic behavior is concerned. This can be deduced from the following important theorem which relates the equilibrium points to the limit surface of a DEM.

Theorem 7. If the admissible stress region bounded by the limit surface associated with a DEM is convex,† then the limit surface is the set of all the equilibrium points corresponding to all possible reference paths.

Proof. It is equivalent to showing that a stress state of a DEM is an equilibrium state if and only if it lies on the limit surface, which is convex, of the model.

Sufficiency. If σ is a stress state of a DEM on the limit surface, then from eqn (31)

$$\sigma = k_u\sigma_0 \quad (\text{for some } \sigma_0 \in \partial\Omega_0). \tag{32}$$

Also, by definition, we have

$$\sigma = \sum_{i=1}^{N} \psi_i\sigma_i = \sum_{i=1}^{N} \psi_i k_i \sigma_0^{(i)} \quad (\sigma_0^{(i)} \in \Omega_0 \quad \forall\, i = 1, \ldots, N). \tag{33}$$

If we rotate the coordinate axes so that the x_1 axis in the stress space is perpendicular to the tangent plane to the yield surface $\partial\Omega_0$ at the point σ_0, as shown in Fig. 3, and define the x_1 coordinate of σ_0, $(\sigma_0)_1$, to be $\alpha, \alpha > 0$, then from eqn (32)

$$(\sigma)_1 = k_u(\sigma_0)_1 = \alpha k_u = \alpha \sum_{i=1}^{N} k_i\psi_i. \tag{34}$$

Since each yield surface is convex and so the region Ω_0 lies completely on one side of any tangent plane of $\partial\Omega_0$, we can deduce

† This is equivalent to the earlier assumption that Ω_0 is convex (if the yield function is homogeneous), which is actually a consequence of Drucker's postulates.

$$(\sigma_0^{(i)})_1 \leqslant \alpha. \tag{35}$$

It follows from eqns (33)–(35) that

$$(\sigma_0^{(i)})_1 = \alpha = (\sigma_0)_1, \quad \forall\ i = 1, \ldots, N,$$

i.e. $\sigma_0^{(i)}$, $\forall\ i = 1, \ldots, N$, lie on the tangent plane $x_1 = \alpha$. Thus, it follows from the shape similarity of the yield surface that all the element stress states are on the associated yield surfaces at the points having the same outward normal direction (perpendicular to the tangent plane). By the principle of normality for each element, the corresponding plastic strain increments of elements are all in the same direction, say **c**, and so is the plastic strain increment of the DEM at σ [following from eqn (15) in Part I]. This shows that the principle of normality holds for a state of the DEM on the limit surface. Now if the total strain increment prescribed is in the direction of **c**, then the plastic strain increment at σ must be zero under the loading condition (otherwise, the stress increment will point outward so that the stress state goes beyond the limit surface), and so therefore is the stress increment. Thus, by definition, the state σ must be an equilibrium point associated with the reference path **c**.

Necessity. If σ is an equilibrium point corresponding to a reference path **c**, then according to Theorem 5, every element state must lie on its associated yield surface at the point corresponding to the outward normal direction **c**. Note that, however, without the assumption of *strict convexity*† of the yield surfaces, we cannot conclude that all element stress states are on a line from the origin (so that by Theorem 6, the model stress state is on the limit surface). Nevertheless, we can still argue as follows. Let R_i denote the subset on a yield surface of constant k_i, in which all points correspond to the same outward normal direction, i.e. R_i lies on a hyperplane in the stress space, which may be described by a linear function in σ_i, so

$$R_i = \{\sigma_i : \hat{F}(\sigma_i) \equiv \sum_{j=1}^{6} a_j(\sigma_i)_j = k_i \quad \text{and} \quad F(\sigma_i, k_i) = 0\}, \tag{36}$$

where j denotes the jth component of a vector, so that the vector gradient $\nabla_\sigma \hat{F}$ is a constant vector throughout the region R_i. Thus, it follows from Theorem 4 that the subset R on the limit surface, corresponding to R_i, can be described by

$$R = \{\sigma : \hat{F}(\sigma) = \sum_{j=1}^{6} a_j(\sigma)_j = k_u \quad \text{and} \quad F(\sigma, k_u) = 0\}. \tag{37}$$

From eqn (22), it follows that

$$\sum_{j=1}^{6} a_j(\sigma)_j = \sum_{j=1}^{6} a_j\left[\sum_{i=1}^{N} \psi_i(\sigma_i)_j\right] = \sum_{i=1}^{N} \psi_i\left[\sum_{j=1}^{6} a_j(\sigma_i)_j\right]. \tag{38}$$

Now if $\sigma_i \in R_i$, then by eqns (36) and (38)

$$\sum_{j=1}^{6} a_j(\sigma)_j = \sum_{i=1}^{N} \psi_i k_i = k_u.$$

Following from eqn (37), we conclude that σ lies in R, which is on the limit surface.

Recall that the existence of equilibrium points has been assured by employing the concept of bounded elastic strain energy. Now we are in a position to address the issue of

† A region Ω and its boundary are said to be strictly convex, if $\partial\Omega$ is convex and there are no two points on $\partial\Omega$ that have the same outward normal direction.

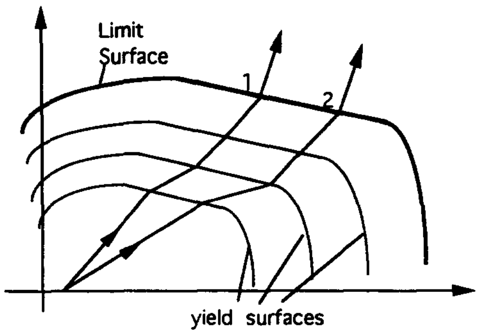

Fig. 4. An illustrative diagram showing non-strict convexity of yield surfaces.

uniqueness of an equilibrium point associated with a reference strain path. This is given as the following theorem.

Theorem 8. *An equilibrium point associated with a reference strain path is uniquely defined* (*regardless of past response history*) *if and only if the admissible stress region bounded by the limit surface is* strictly *convex.*

The proof of Theorem 8 can be done simply by considering the schematic diagram shown in Fig. 4, where the yield surfaces (and the associated limit surface) are not strictly convex. Given a uni-directional strain path following different previous histories, we may end up with different equilibrium points as points 1 and 2 shown in the figure. If, instead, the admissible stress region is strictly convex, then corresponding to a reference strain path, there is only one point on the limit surface that has the outward normal in that direction. Thus, following the flow rule based on the principle of normality, the equilibrium point is uniquely defined.

With the theorems presented above, we may now investigate in detail the property of erasure-of-memory that is exhibited by real materials (cf. Part I). This property can be stated as follows: If a material has been stabilized by "out-of-phase" cycling, i.e. loading with non-proportional strain cycles,† and if the subsequent strain paths remain in the region enclosed by the out-of-phase cycling, then one "big" strain cycle, which is sufficiently smooth and long so that there exists at least one equilibrium state associated with it, will always bring the material back to the particular equilibrium state associated with that big strain cycle. This property is very useful in conducting experiments on cyclic plasticity (Lamba and Sidebottom, 1978a), since a single specimen can always be brought back to the same reference state, and so can be used repeatedly in characterizing material response to various loading paths. This ensures that more reliable results can be obtained with considerably less labor and cost.

It may be deduced that the existence and uniqueness of equilibrium points associated with different reference paths are the necessary and sufficient conditions for a DEM to exhibit the property of erasure-of-memory, since then every time a "big" smooth strain cycle is prescribed, the system will be brought back to the particular equilibrium states associated with that strain cycle, regardless of what the previous history is. This leads to the following important theorem.

Theorem 9. *A DEM possesses the property of erasure-of-memory if and only if its admissible stress region bounded by the associated limit surface is bounded and strictly convex, from which the existence and uniqueness of equilibrium points follow.*

†Experimental results have shown that the uniaxial peak stress resulting from out-of-phase hardening is about 40% higher than that from uniaxial cycling (Lamba and Sidebottom, 1978a). If a material has not yet been out-of-phase stabilized, its yield condition becomes variant and depends on the non-proportionality of the loading path. This phenomenon cannot be characterized by conventional plasticity models unless special treatment is made (Sugiura *et al.*, 1987).

In summary, if the yield functions used in the definition of a DEM is homogeneous and *strictly quasi-convex*† so that the limit surface exists and forms a strictly convex region, then the DEM can exhibit the property of erasure-of-memory. Actually, as can be deduced, the conditions stated in Theorem 8 serve as the general criteria for any plasticity model to demonstrate the property of erasure-of-memory that real materials have.

4. CONCLUSIONS

A general theory based on ideal plasticity and geometrical considerations of the invariant yield surfaces associated with the new DEMs is presented to elucidate important properties of material behavior in general plasticity. It is shown that the property of erasure-of-memory exhibited by real materials is a consequence of the existence and uniqueness of equilibrium states corresponding to all possible reference paths, which in turn results from the property that the admissible stress region bounded by the limit surface associated with a Distributed-Element Model is bounded and strictly convex. Establishment of the theorems presented in this study provides us with clear insight into the elastic–plastic response mechanisms of real materials under complicated cyclic loading conditions, which surely helps further studies on the related subjects of general plasticity.

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†Mathematically, it can be shown (Franchi *et al.*, 1990) that if a yield function is strictly quasi-convex, then the associated yield surface forms a strictly convex region. A scalar function $F(\boldsymbol{\sigma})$ is strictly quasi-convex at $\boldsymbol{\sigma}_1$ if

$$F(\boldsymbol{\sigma}_2) < F(\boldsymbol{\sigma}_1) \Rightarrow F(\boldsymbol{\sigma}) < F(\boldsymbol{\sigma}_1), \quad \forall\ \boldsymbol{\sigma} = \alpha\boldsymbol{\sigma}_1 + (1-\alpha)\boldsymbol{\sigma}_2, \quad 0 < \alpha < 1.$$